

Maxwell Times in Higher-Order Generalized Hydrodynamics: Classical Fluids, and Carriers and Phonons in Semiconductors

Clóves G. Rodrigues¹, Carlos A. B. Silva², José G. Ramos³, Roberto Luzzi^{3*}

¹*Departamento de Física, Pontifícia Universidade
Católica de Goiás, 74605-010 Goiânia, Goiás, Brazil*

²*Instituto Tecnológico de Aeronáutica,
12228-901 São José dos Campos, SP, Brazil*

³*Condensed Matter Physics Department,
Institute of Physics “Gleb Wataghin” State University
of Campinas-Unicamp, 13083-859 Campinas, SP, Brazil*

(Dated: July 2, 2015)

Abstract

A family of the so-called Maxwell times which arises in the context of Higher-Order Generalized Hydrodynamics (also called Mesoscopic Hydro-Thermodynamics) is evidenced. This is done in the framework of a HOGH build within a statistical foundation in terms of a Non-Equilibrium Statistical Ensemble Formalism. It consists in a description in terms of the densities of particles and energy and their fluxes of all orders, with the motion described by a set of coupled nonlinear integro-differential equations involving them. These Maxwell Times have a fundamental role in determining the type of hydrodynamic motion that the system would display in the given condition and constraints. The different types of motion are well described by contractions of the full description done in terms of a reduced number of fluxes up to a certain order.

PACS numbers: 67.10.Jn; 05.70.Ln; 68.65.-k; 81.05.Ea

* group home page: www.ifi.unicamp.br/aurea; email: cloves@pucgoias.edu.br

I. INTRODUCTION

We present a generalization of Maxwell time [1], which can also be referred-to as generalized Maxwellian relaxation time [2], a family of which is present in the evolution equations of Higher-Order Hydrodynamics, also dubbed as Mesoscopic Hydro-Thermodynamics (MHT) [3-6]. Such generalized Maxwell times are associated to the dampening of densities of particles (molecules in a classical fluid) or quasi-particles (Bloch-band electrons, harmonic phonons, magnons, polaritons, etc., in solid state matter, in semiconductors, for example), and of their energy density, together with the fluxes of all orders of both.

The origin of Maxwell times goes back to the fundamental article by J. C. Maxwell in 1867 on the dynamical theory of gases [1], in what is related to viscoelasticity and which is being expressed as the quotient of the dynamical viscosity coefficient with the modulus of rigidity. According to Maxwell it may be called the "time of relaxation" of the elastic force. It has been noticed [2] that given a fluid subjected to some variable external forces, which vary periodically in time with frequency ω , of the period $1/\omega$ is large compared with Maxwell time, i.e. $\omega\theta \ll 1$, the fluid under consideration will be have as an ordinary viscous fluid. If however, the frequency ω is sufficiently large (so that $\omega\theta \gg 1$, the fluid will be have as an amorphous solid.

We present a detailed study of the generalized Maxwell times that arise in a Mesoscopic Hydro-Thermodynamics, built upon a Non-Equilibrium Statistical Ensemble Formalism (NESEF for short) [7-10], at the classical and quantum mechanical levels of HOGH described in Refs. [5,6,11,12].

In Section II is considered the classical MHT of a fluid of particles embedded in a thermal bath. In Section III the case of a MHT at the quantum level of a system of phonons in semiconductors, and in Section IV a MHT at the quantum level of a system of electrons in doped semiconductors.

II. MAXWELL TIMES IN THE MHT OF A CLASSICAL FLUID

Construction of MHT in the framework of a Nonequilibrium Statistical Ensemble Formalism is presented in Ref. [5]. There it is provided an approach enabling for the coupling and simultaneous treatment of the kinetic and hydrodynamic levels of descriptions. It is based on a complete thermo-statistical approach in terms of the densities of matter and energy and their fluxes of all orders, as well as on their direct and cross correlations, covering systems arbitrarily far-removed from equilibrium. The set of coupled nonlinear integro-differential hydrodynamic equations is derived. They are the evolution equations of a Grad-type approach involving the moments of all orders of the single-particle distribution $f_1(\mathbf{r}, \mathbf{p}; t)$, derived from a generalized kinetic equation, built in the framework of a Nonequilibrium Statistical Ensemble Formalism [13].

The moments of the single-particle distribution function, in momentum space \mathbf{p} , are the hydrodynamic variables

$$n(\mathbf{r}, t) = \int d^3p f_1(\mathbf{r}, \mathbf{p}; t), \quad (1)$$

which is the densities of particles,

$$\mathbf{I}_n(\mathbf{r}, t) = \int d^3p \mathbf{u}(\mathbf{p}) f_1(\mathbf{r}, \mathbf{p}; t), \quad (2)$$

with

$$\mathbf{u}(\mathbf{p}) = \frac{\mathbf{p}}{m}, \quad (3)$$

where \mathbf{I}_n is the flux (current) of particles,

$$I_n^{[2]}(\mathbf{r}, t) = \int d^3p \mathbf{u}^{[2]}(\mathbf{p}) f_1(\mathbf{r}, \mathbf{p}; t), \quad (4)$$

where $\mathbf{u}^{[2]} = [\mathbf{u} : \mathbf{u}]$ is the inner tensorial product of vectors \mathbf{u} , with $I_n^{[2]}$ being the second-order flux (or flux of the flux), a rank-2 tensor, which multiplied by the mass is related to the pressure tensor, and

$$I_n^{[\ell]}(\mathbf{r}, t) = \int d^3p \mathbf{u}^{[\ell]}(\mathbf{p}) f_1(\mathbf{r}, \mathbf{p}; t), \quad (5)$$

are the higher-order fluxes of order $\ell \geq 3$ (the previous three of Eqs. (1), (2) and (4), are those for $\ell = 0, 1$ and 2 respectively) where $\mathbf{u}^{[\ell]}$ is the ℓ -rank tensor consisting of the inner tensorial product of ℓ vectors \mathbf{u} of Eq. (3), that is,

$$\mathbf{u}^{[\ell]}(\mathbf{p}) = \left[\frac{\mathbf{p}}{m} : \frac{\mathbf{p}}{m} \dots (\ell\text{-times}) \dots : \frac{\mathbf{p}}{m} \right]. \quad (6)$$

We do have what can be called the *family of hydrodynamical variables describing the material motion*, i.e., the set

$$\{n(\mathbf{r}, t); \mathbf{I}_n(\mathbf{r}, t); \{I_n^{[\ell]}(\mathbf{r}, t)\}\}, \quad (7)$$

with $\ell = 2, 3, \dots$, which we call the MHT-family n .

On the other hand, we do have the *family of hydrodynamical variables describing the thermal motion*, which we call the MHT-family h , consisting of

$$h(\mathbf{r}, t) = \int d^3p \frac{p^2}{2m} f_1(\mathbf{r}, \mathbf{p}; t), \quad (8)$$

$$\mathbf{I}_h(\mathbf{r}, t) = \int d^3p \frac{p^2}{2m} \frac{\mathbf{p}}{m} f_1(\mathbf{r}, \mathbf{p}; t), \quad (9)$$

$$I_h^{[\ell]}(\mathbf{r}, t) = \int d^3p \frac{p^2}{2m} \mathbf{u}^{[\ell]}(\mathbf{p}) f_1(\mathbf{r}, \mathbf{p}; t), \quad (10)$$

with $\ell = 2, 3, \dots$, that is, in compact form, those in the set

$$\{h(\mathbf{r}, t); \mathbf{I}_h(\mathbf{r}, t); \{I_h^{[\ell]}(\mathbf{r}, t)\}\}, \quad (11)$$

which are, respectively, the density of energy, its first vectorial flux (heat current), and the higher-order tensorial fluxes. It can be noticed that in this case of a parabolic type energy momentum dispersion relation, $E(p) = p^2/2m$, the set of Eq. (11) is encompassed in the previous one: in fact

$$h(\mathbf{r}, t) = \frac{m}{2} Tr \{I_n^{[2]}(\mathbf{r}, t)\}, \quad (12)$$

$$\mathbf{I}_h(\mathbf{r}, t) = \frac{m}{2} Tr_2 \{I_n^{[3]}(\mathbf{r}, t)\}, \quad (13)$$

where Tr_2 stands for the contraction of the first two indexes, and, in general

$$I_h^{[\ell]}(\mathbf{r}, t) = \frac{m}{2} Tr_2 \{I_n^{[\ell+2]}(\mathbf{r}, t)\}, \quad (14)$$

for all the other higher-order fluxes of energy, that is, any flux of energy of order ℓ is contained in the flux of matter of order $\ell + 2$.

Let us consider the equations of evolutions for the basic macrovariables of the family of particle motion, which for the general flux of order ℓ ($\ell = 0, 1, 2, \dots$) is

$$\frac{\partial}{\partial t} I_n^{[\ell]}(\mathbf{r}, t) = \int d^3p \mathbf{u}^{[\ell]}(\mathbf{p}) \frac{\partial}{\partial t} f_1(\mathbf{r}, \mathbf{p}; t), \quad (15)$$

Using in Eq. (15) the evolution equations for $f_1(\mathbf{r}, \mathbf{p}; t)$ of Ref. [13] it follows the general set of coupled equations for the density, $\ell = 0$, the current, $\ell = 1$, and all the other higher-order fluxes, $\ell \geq 2$, given by

$$\begin{aligned} & \frac{\partial}{\partial t} I_n^{[\ell]}(\mathbf{r}, t) + \nabla \cdot I_n^{[\ell+1]}(\mathbf{r}, t) = \\ & = -\frac{1}{m} \sum_{s=1}^{\ell} \sigma(1, s) [\mathcal{F}(\mathbf{r}, t) I_n^{[\ell-1]}(\mathbf{r}, t)] - \theta_{n\ell}^{-1} I_n^{[\ell]}(\mathbf{r}, t) + \\ & a_{L0} \sum_{s=1}^{\ell} \sigma(1, s) [\nabla I_n^{[\ell-1]}(\mathbf{r}, t)] + 2\ell a_{L1} \nabla \cdot I_n^{[\ell+1]}(\mathbf{r}, t) + \\ & J_{NL}^{[\ell]}(\mathbf{r}, t) + S_n^{[\ell]}(\mathbf{r}, t). \end{aligned} \quad (16)$$

The last term on the right of Eq. (16) is given by

$$\begin{aligned} S_n^{[\ell]}(\mathbf{r}, t) = & b_{\tau 0} \{ \sigma(1, s) [1^{[2]} I_n^{[\ell-2]}(\mathbf{r}, t)] \} + b_{\tau 1} \frac{2}{m} \{ \sigma(1, s) [1^{[2]} I_h^{[\ell-2]}(\mathbf{r}, t)] \} \\ & + 3\ell a_{\tau 1} \frac{2}{m} I_h^{[\ell]}(\mathbf{r}, t) + a_{L1} \frac{2}{m} \sum_{s=1}^{\ell} \sigma(1, s) [\nabla I_h^{[\ell-1]}(\mathbf{r}, t)] + R_n^{[\ell]}(\mathbf{r}, t), \end{aligned} \quad (17)$$

with all quantities involved given in Ref. [5]: we omit the details that are not necessary for the analysis of Maxwell times.

The first contribution on the right of Eq. (17) contain the flux of particles of order $\ell - 2$, whereas the other three contributions are associated to the fluxes of energy of orders $\ell - 2$, $\ell - 1$ and ℓ , terms that can be considered as associated to thermo-striction effects which couple these equations with the set of kinetic equations describing the motion. However, it can be noticed the already mentioned fact that the fluxes of energy can be given in terms of those of particles, as given in Eqs. (12) to (14).

The second term on the right of Eq. (16), for the evolution of the flux of order ℓ , is the one that contains the Maxwell time $\theta_{n\ell}$ associated to such flux, namely

$$\theta_{n\ell}^{-1} = -\ell |a_{\tau 0}| + \ell(\ell - 1) |b_{\tau 1}|, \quad (18)$$

with the coefficients $a_{\tau 0}$ and $b_{\tau 1}$ given by

$$a_{\tau 0} = \frac{\mathcal{V}}{(2\pi)^3} \frac{4\pi}{3} \int dQ Q^4 f_{\tau 0}(Q), \quad (19)$$

with

$$f_{\tau 0}(Q) = -\frac{n_R M \beta_0^{3/2} \pi}{\mathcal{V} \sqrt{2\pi} m^2} \frac{|\psi(Q)|^2}{Q} \left(\frac{m}{M} + 1 \right), \quad (20)$$

where $\psi(Q)$ is the Fourier transform of the potential energy $w(|\mathbf{r}_j - \mathbf{R}_\mu|)$, between the j -th particle at position \mathbf{r} and the one in the thermal bath at position \mathbf{R}_μ , n_R is the density of particles in the thermal bath, \mathcal{V} is the volume, and $\beta_0^{-1} = k_B T_0$, with T_0 being the temperature of the thermal bath, moreover

$$b_{\tau 1} = -\frac{a_{\tau 0}}{5} \left(\frac{m}{M} + 1 \right)^{-1}, \quad (21)$$

and m and M are the mass of the particles and of the particles in the thermal bath respectively. According to Eqs. (18) and (21) we can write

$$\theta_{n\ell}^{-1} = -\ell |a_{\tau 0}| \left[1 + \frac{1}{5}(\ell - 1) \left(\frac{M}{m + M} \right) \right], \quad (22)$$

and it can be noticed that $|a_{\tau 0}|$ is the reciprocal of the Maxwell time of the first flux which is the reciprocal of the relaxation time of the linear momentum $\mathbf{p}(\mathbf{r}, t)$, once $\mathbf{p}(\mathbf{r}, t) = m\mathbf{I}_n(\mathbf{r}, t)$.

Maxwell times of Eq. (18) have their origin from the collision integral $J_\tau^{(2)}(\mathbf{r}, \mathbf{p}; t)$ in the kinetic equation for the single-particle distribution present in Eq. (A.11) in Appendix A in Ref. [13]. It arises out of the interaction with the thermal bath in the contributions called effective friction force and diffusion in momentum space.

In a recurrent procedure it follows from Eq. (18) that we can write

$$\theta_{n\ell}^{-1} = \ell \left[1 + \frac{M}{5(m + M)}(\ell - 1) \right] \theta_{n1}^{-1}, \quad (23)$$

what tell us that any characteristic time for $\ell \geq 2$ is proportional to the one of $\ell = 1$, that is, the one for the first flux which, as notice above, multiplied by the mass m is the linear momentum density and then all are proportional to the linear momentum relaxation time. On the other hand we do have that

$$\frac{\theta_{n_{\ell+1}}}{\theta_{n_\ell}} = \frac{\ell}{\ell + 1} \frac{5(1 + x) + \ell - 1}{5(1 + x) + \ell}, \quad (24)$$

for $\ell = 1, 2, 3, \dots$ and where $x = m/M$, then the ordered sequence

$$\theta_{n_1} > \theta_{n_2} > \theta_{n_3} \dots > \theta_{n_\ell} > \theta_{n_{\ell+1}} > \dots, \quad (25)$$

is verified, and it can be seen that $\theta_{n_\ell} \rightarrow 0$ as $\ell \rightarrow \infty$. Moreover, according to Eq. (23) it follows that

$$\theta_{n_\ell} = \frac{5(1 + x)}{\ell[5(1 + x) + \ell - 1]} \theta_{n_1}, \quad (26)$$

Comparing with the second flux ($\ell = 2$) which is the one related to the pressure tensor, it follows that for Brownian particles ($x \gg 1$) $\theta_{n_2} \simeq \theta_{n_1}/2$ and for Lorentz particles ($x \ll 1$) $\theta_{n_2} \simeq 5\theta_{n_1}/12$. A comparison with the third flux leads to the results that $\theta_{n_3} \simeq \theta_{n_1}/3$ and $\theta_{n_3} \simeq 5\theta_{n_1}/21$ for the Brownian and Lorentz particles respectively. For any ℓ we do have approximately:

1) for the Brownian particle ($m/M \gg 1$)

$$\theta_{n_\ell} \simeq \frac{\theta_{n_1}}{\ell} , \quad (27)$$

2) for the Lorentz particle ($m/M \ll 1$)

$$\theta_{n_\ell} \simeq \frac{5}{(4 + \ell)\ell} \theta_{n_1} , \quad (28)$$

or $\theta_{n_\ell} \simeq 5\theta_{n_1}/\ell^2$ for large ℓ .

According to Eq. (24) as the order of the flux largely increases its Maxwell characteristic time approaches zero, and $\theta_{n_{\ell+1}}/\theta_{n_\ell} \simeq 1$, with both practically null. In Fig. 1 it is displayed the ratio of some characteristic times compared with the one of the linear momentum in terms of the ratio m/M .

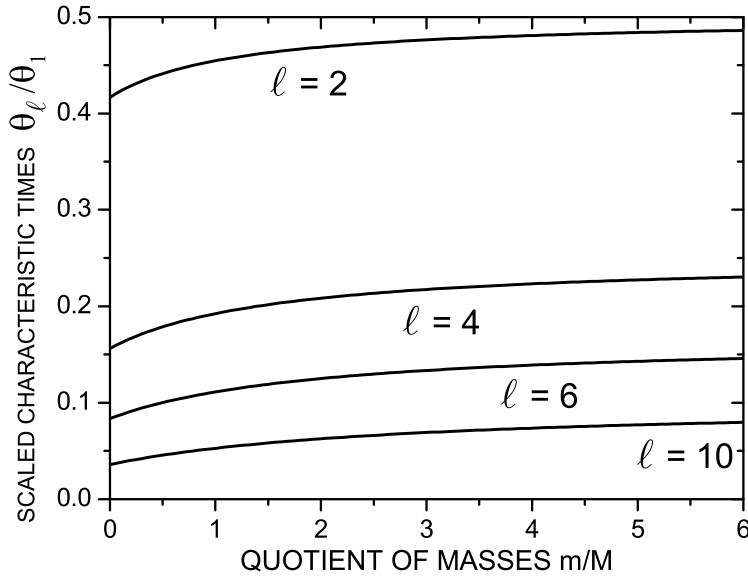


FIG. 1: The quotient between Maxwell characteristic times and the one of the first flux as a function of $x = m/M$.

We emphasized that the hierarchy shown in Eq. (25) is evidencing the important fact of the ever decreasing value of the Maxwell times with the increasing order of the fluxes. This

is quite relevant for the question of how to introduce a contracted description (i.e., keeping a reduced number of fluxes) for characterizing the particular type of hydrodynamic motion (diffusion, damped waves, etc.) that the system displays, as shown later on.

III. MAXWELL TIMES IN THERMAL TRANSPORT

In this section we concentrate the attention on the evolution of the thermal quantities

$$\{h(\mathbf{r}, t); \mathbf{I}_h(\mathbf{r}, t); \{I_h^{[\ell]}(\mathbf{r}, t)\}\}, \quad (29)$$

with $\ell = 2, 3, \dots$, that is, the density of energy and its fluxes of all orders given in Eq. (11). Their evolution equations are

$$\frac{\partial}{\partial t} I_h^{[\ell]}(\mathbf{r}, t) = \int d^3p \frac{p^2}{2m} \mathbf{u}^{[\ell]}(\mathbf{p}) \frac{\partial}{\partial t} f_1(\mathbf{r}, \mathbf{p}; t), \quad (30)$$

in which we need to introduce the kinetic equation for the single particle $f_1(\mathbf{r}, \mathbf{p}; t)$ which is given in Ref. [13]. Performing the lengthy calculations involved we finally arrive at the general evolution equations ($\ell = 0, 1, 2, \dots$)

$$\begin{aligned} & \frac{\partial}{\partial t} I_h^{[\ell]}(\mathbf{r}, t) + \nabla \cdot I_h^{[\ell+1]}(\mathbf{r}, t) = \\ & = -\mathcal{F}(\mathbf{r}, t) I_n^{[\ell+1]}(\mathbf{r}, t) - \sum_{s=1}^{\ell} \sigma(1, s) [\mathcal{F}(\mathbf{r}, t) I_h^{[\ell-1]}(\mathbf{r}, t)] + \theta_{h_\ell}^{-1} I_h^{[\ell]}(\mathbf{r}, t) + \\ & a_{L0} \sum_{s=1}^{\ell} \sigma(1, s) [\nabla I_n^{[\ell-1]}(\mathbf{r}, t)] + 2(\ell + 3) a_{L1} \nabla \cdot I_h^{[\ell+1]}(\mathbf{r}, t) + \\ & S_h^{[\ell]}(\mathbf{r}, t), \end{aligned} \quad (31)$$

where

$$\theta_{h_\ell}^{-1} = (\ell + 2) |a_{\tau 0}| + [\ell^2 + 5(\ell + 1)] |b_{\tau 1}|, \quad (32)$$

is the reciprocal of the Maxwell time associated to the ℓ -th order flux of energy. Moreover,

$$\begin{aligned} S_h^{[\ell]}(\mathbf{r}, t) = & (2\ell + 3) b_{\tau 0} \frac{m}{2} I_n^{[\ell]}(\mathbf{r}, t) + \\ & b_{\tau 0} \{ \hat{\sigma}_\ell [1^{[2]} I_h^{[\ell-2]}(\mathbf{r}, t)] \} + b_{\tau 1} \frac{2}{m} \{ \hat{\sigma}_\ell [1^{[2]} I_{h2}^{[\ell-2]}(\mathbf{r}, t)] \} + \\ & a_{L0} m \nabla \cdot I_n^{[\ell+1]}(\mathbf{r}, t) + \\ & a_{L1} \frac{2}{m} \sum_{s=1}^{\ell} \sigma(1, s) [\nabla I_{h2}^{[\ell-1]}(\mathbf{r}, t)] + \\ & 3(\ell + 2) a_{\tau 1} \frac{2}{m} I_{h2}^{[\ell]}(\mathbf{r}, t) + \\ & R_h^{[\ell]}(\mathbf{r}, t), \end{aligned} \quad (33)$$

is a contribution which couples the thermal motion to the material motion (thermal-striction effect), $R_h^{[\ell]}(\mathbf{r}, t)$ contains contribution of all the other higher-order fluxes ($\geq \ell + 2$). The several kinetic coefficients are presented in Appendix B in Ref. [13].

Taking into account that

$$|b_{\tau 1}| = \frac{1}{5(1+x)} |a_{\tau 0}|, \quad (34)$$

it follows that Maxwell times have the property that

$$\frac{\theta_{h_{\ell+1}}}{\theta_{h_\ell}} = \frac{5(\ell+2)(1+x) + \ell^2 + 5(\ell+1)}{5(\ell+3)(1+x) + (\ell+1)^2 + 5(\ell+2)} < 1, \quad (35)$$

for $\ell = 0, 1, 2, 3, \dots$, and where $x = m/M$. The ordered sequence

$$\theta_{h_0} > \theta_{h_1} > \theta_{h_2} > \theta_{h_3} > \dots > \theta_{h_\ell} > \theta_{h_{\ell+1}} > \dots \quad (36)$$

is verified in an analogous condition on the Maxwell times associated to the particle motion. It may be noticed that for large ℓ , θ_ℓ goes as ℓ^2 , i.e., going to zero as $\ell \rightarrow \infty$.

For the Brownian particle ($x \gg 1$) the ratio in Eq. (35) tends asymptotically to $(\ell+2)/(\ell+3)$ while for Lorentz particles ($x \ll 1$) the ratio tends to $[\ell^2 + 5(2\ell+3)]/[(\ell+1)^2 + 5(2\ell+5)]$. Considering that the Maxwell time of the first flux of mater, θ_{n_1} is $|a_{\tau 0}|^{-1}$ (cf. Eq. (22)) we can establish the relation between it and the Maxwell time for the energy

$$\theta_{h_0} = \frac{1+x}{2x+3} \theta_{n_1}, \quad (37)$$

what tell us that $\theta_{h_0} < \theta_{n_1}$. For a Brownian particle $\theta_{h_0} = \theta_{n_1}/2$ and for a Lorentz particle $\theta_{h_0} = \theta_{n_1}/3$. Moreover, for general ℓ we have that

$$\theta_{h_\ell} = \frac{5(1+x)}{5(\ell+2)(1+x) + \ell^2 + 5(\ell+1)} \theta_{n_1}, \quad (38)$$

Therefore, recalling that the first flux of matter multiplied by the particle mass is the linear momentum density, as a general rule we can state that the energy relaxation time is smaller than momentum relaxation time. As already notice at the end of the previous Section, the hierarchy shown in Eq. (36) is quite relevant for the choose of the contracted description to be used.

We proceed next to consider the case of a MHT at a quantum mechanical level for the cases of phonons and electrons in semiconductors.

IV. MAXWELL TIMES IN MHT OF PHONONS IN SEMICONDUCTORS

We consider a system of longitudinal acoustic LA phonons in a semiconductor in anharmonic interaction with the accompanying transverse acoustic TA phonons. The sample is in contact with a thermostat at temperature T_0 . An external pumping source drives the LA phonon system out of equilibrium. The system is characterized at the microscopic level by the Hamiltonian

$$\hat{H} = \hat{H}_{OS} + \hat{H}_{OB} + \hat{H}_{SB} + \hat{H}_{SP}, \quad (39)$$

which consists of the Hamiltonian of the free LA phonons

$$\hat{H}_{OS} = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} (a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} + 1/2), \quad (40)$$

where $\omega_{\mathbf{q}}$ is the frequency dispersion relation and the sum on \mathbf{q} runs over the Brillouin zone, and the one of the TA phonons, which we call the thermal bath in which the LA phonons are embedded, given by

$$\hat{H}_{OB} = \sum_{\mathbf{q}} \hbar \Omega_{\mathbf{q}} (b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}} + 1/2), \quad (41)$$

where $\Omega_{\mathbf{q}}$ is the frequency dispersion relation; $a_{\mathbf{q}}^{\dagger}(a_{\mathbf{q}})$ and $b_{\mathbf{q}}^{\dagger}(b_{\mathbf{q}})$ are the corresponding creation (annihilation) operators in mode \mathbf{q} .

Moreover, the interaction of the LA phonons with the thermal bath is given by

$$\hat{H}_{SB} = \sum_{\mathbf{k}, \mathbf{q}} M_{\mathbf{k}, \mathbf{q}} a_{\mathbf{q}} b_{\mathbf{k}+\mathbf{q}}^{\dagger} b_{-\mathbf{k}}^{\dagger} + \text{hc}, \quad (42)$$

where we have retained the only process that contributes to the kinetic equations ($\text{LA} \rightleftharpoons \text{TA} + \text{TA}$) and we have neglected nonlinear contributions; $M_{\mathbf{k}, \mathbf{q}}$ accounts for the coupling strengths. Finally \hat{H}_{SP} is the interaction energy operator for the phonons and an external pumping source, to be specified in each case.

At the macroscopic level (the nonequilibrium thermodynamic state) NESEF requires first to specify the basic variables that are to be used to characterize the nonequilibrium ensemble [7-10]. A priori, when the system is initially driven away from equilibrium, it is necessary to include all the observables of the system what is attained introducing many-particle dynamical operators [14,15]; in the present case it suffices to take only the single-phonon dynamical operators $\hat{\nu}_{\mathbf{q}, \mathbf{Q}} = a_{\mathbf{q}+\mathbf{Q}/2}^{\dagger} a_{\mathbf{q}-\mathbf{Q}/2}$ in the second-quantization representation in reciprocal space. The two-phonon dynamical operator and higher-order ones can be

ignored because of Bogoliubov's principle of correlation weakening [15]. Moreover, it would be necessary to also include the amplitudes $a_{\mathbf{q}}^\dagger$ and $a_{\mathbf{q}}$ because their eigenstates are the coherent states [16], and the phonon pairs [17], both are disregarded because they are of no practical relevance for the problem considered here. Another basic microdynamical variable that needs be incorporated is the energy of the bath, and then the basic set is composed of

$$\{\hat{\nu}_{\mathbf{q}}, \hat{\nu}_{\mathbf{q},\mathbf{Q}}, \hat{H}_{OB}\}, \quad (43)$$

with $\mathbf{Q} \neq 0$, where we have separated out the so-called populations, $\hat{\nu}_{\mathbf{q}} = a_{\mathbf{q}}^\dagger a_{\mathbf{q}}$, from those with $\mathbf{Q} \neq 0$ which are related to the change in space of the populations (they are also called coherences [18]).

Therefore, for the present case the nonequilibrium statistical operator is

$$\varrho_\epsilon(t) = \exp\left\{-\hat{S}(t, 0) + \int_{-\infty}^t dt' e^{\epsilon(t'-t)} \frac{d}{dt'} \hat{S}(t', t' - t)\right\}, \quad (44)$$

where

$$\begin{aligned} \hat{S}(t, 0) &= -\ln \bar{\varrho}(t, 0) = \\ &= \phi(t) + \sum_{\mathbf{q}} F_{\mathbf{q}}(t) a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \sum_{\mathbf{q}, \mathbf{Q} \neq 0} F_{\mathbf{q}, \mathbf{Q}}(t) a_{\mathbf{q}+\mathbf{Q}/2}^\dagger a_{\mathbf{q}-\mathbf{Q}/2} + \beta_0 \hat{H}_{OB}, \end{aligned} \quad (45)$$

$$\hat{S}(t', t' - t) = \exp\left\{-\frac{1}{i\hbar}(t' - t)\hat{H}\right\} \hat{S}(t', 0) \exp\left\{\frac{1}{i\hbar}(t' - t)\hat{H}\right\}, \quad (46)$$

\hat{H}_{OB} is given in Eq. (41), $\phi(t) = \ln \bar{Z}(t)$ plays the role of a logarithm of a nonequilibrium partition function $\bar{Z}(t)$, and $\beta_0 = 1/(k_B T_0)$.

The average values of the microdynamical variables in set (43) over the nonequilibrium ensemble provide the variables that characterized the nonequilibrium macroscopic state of the system, which we indicate by

$$\{\nu_{\mathbf{q}}(t), \nu_{\mathbf{q},\mathbf{Q}}(t), E_B\}, \quad (47)$$

and the nonequilibrium thermodynamic variables conjugated to them are

$$\{F_{\mathbf{q}}(t), F_{\mathbf{q},\mathbf{Q}}(t), \beta_0\}. \quad (48)$$

Going over to direct space (anti-transforming Fourier in variable \mathbf{Q}) we obtain the space and crystalline momentum-dependent distribution function $\nu_{\mathbf{q}}(\mathbf{r}, t)$, in terms of which is

built the phonon Higher-Order Generalized Hydro-Thermodynamics which consists of two families of hydrodynamical variables, namely, the one associated to the quasi-particles (the phonons) motion (we call it the n -family)

$$\{n(\mathbf{r}, t), \mathbf{I}_n(\mathbf{r}, t), \{I_n^{[\ell]}(\mathbf{r}, t)\}\}, \quad (49)$$

where $\ell = 2, 3, \dots$, and $n(\mathbf{r}, t)$ stands for the number of phonons at time t in position \mathbf{r} , namely

$$n(\mathbf{r}, t) = \sum_{\mathbf{q}} \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (50)$$

the first flux of this quantity

$$\mathbf{I}_n(\mathbf{r}, t) = \sum_{\mathbf{q}} \nabla_{\mathbf{q}} \omega_{\mathbf{q}} \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (51)$$

and the higher-order fluxes

$$I_n^{[\ell]}(\mathbf{r}, t) = \sum_{\mathbf{q}} u^{[\ell]}(\mathbf{q}) \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (52)$$

with $\ell = 2, 3, \dots$, defining the ℓ -order flux (ℓ -rank tensor) where

$$u^{[\ell]}(\mathbf{q}) = [\nabla_{\mathbf{q}} \omega_{\mathbf{q}} : \dots \ell - times \dots : \nabla_{\mathbf{q}} \omega_{\mathbf{q}}], \quad (53)$$

is a ℓ -rank tensor consisting of the tensorial inner product of ℓ -times the group velocity of the \mathbf{q} -mode phonon, $\nabla_{\mathbf{q}} \omega_{\mathbf{q}}$.

On the other hand we do have the family associated to the energy motion (heat transport, and we call it the h -family)

$$\{h(\mathbf{r}, t), \mathbf{I}_h(\mathbf{r}, t), \{I_h^{[\ell]}(\mathbf{r}, t)\}\}, \quad (54)$$

where

$$h(\mathbf{r}, t) = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (55)$$

$$\mathbf{I}_h(\mathbf{r}, t) = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} \nabla_{\mathbf{q}} \omega_{\mathbf{q}} \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (56)$$

$$I_h^{[\ell]}(\mathbf{r}, t) = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} u^{[\ell]}(\mathbf{q}) \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (57)$$

which are, respectively, the energy density, its first (vectorial) flux, and the higher-order ($\ell = 2, 3, \dots$) tensorial fluxes at time t in position \mathbf{r} .

Consequently, the hydrodynamic equations of motion (evolution equations for the quantities above) are

$$\frac{\partial}{\partial t} I_p^{[\ell]}(\mathbf{r}, t) = \sum_{\mathbf{q}} K_p^{[\ell]}(\mathbf{q}) \frac{\partial}{\partial t} \nu_{\mathbf{q}}(\mathbf{r}, t), \quad (58)$$

where, for $p \equiv n$, $K_n^{[\ell]}(\mathbf{q}) = u^{[\ell]}(\mathbf{q})$ and for $p \equiv h$, $K_h^{[\ell]}(\mathbf{q}) = \hbar \omega_{\mathbf{q}} u^{[\ell]}(\mathbf{q})$, recalling that $\ell = 0$ corresponds to the densities, $\ell = 1$ to the vectorial fluxes, and $\ell \geq 2$ for the higher-order fluxes.

Evidently, all the evolution equations in Eq. (58) are determined by the unique equation of motion for the single-phonon distribution function $\nu_{\mathbf{q}}(\mathbf{r}, t)$. It follows from the evolution equation which results in taking the average over the nonequilibrium ensemble of the quantum mechanical Heisenberg equation of motion for the microdynamical variable $\hat{\nu}_{\mathbf{q}\mathbf{Q}} = a_{\mathbf{q}+\mathbf{Q}/2}^\dagger a_{\mathbf{q}-\mathbf{Q}/2}$, (for practical convenience is calculated in reciprocal space), that is

$$\frac{\partial}{\partial t} \nu_{\mathbf{q},\mathbf{Q}}(t) = \text{Tr} \left\{ \frac{1}{i\hbar} [\hat{\nu}_{\mathbf{q}\mathbf{Q}}, \hat{H}] \varrho_\epsilon(t) \times \varrho_B \right\}. \quad (59)$$

where $\varrho_\epsilon(t)$ is the nonequilibrium statistical operator of Eq. (44), and ϱ_B is the canonical statistical distribution of the thermal bath (the TA phonons) at the temperature T_0 . Performing such average is extremely difficult and then it is necessary to resort to the introduction of a practical nonlinear quantum kinetic theory [7-10,19,20], which, for the present case, can be used in the approximation consisting in retaining in the collision integral contributions up to second order in the interaction strength (Markovian approximation [9,19,21]). In that approximation it follows, after going over to direct space, a generalization of the so-called Peierls-Boltzmann equation given by [22]

$$\begin{aligned} \frac{\partial}{\partial t} \nu_{\mathbf{q}}(\mathbf{r}, t) = & -\nabla \cdot \nabla_{\mathbf{q}} \tilde{\omega}_{\mathbf{q}} \nu_{\mathbf{q}}(\mathbf{r}, t) - \Gamma_{\mathbf{q}} [\nu_{\mathbf{q}}(\mathbf{r}, t) - \nu_{\mathbf{q}}^0] \\ & + J_{\mathbf{q}S}(\mathbf{r}, t), \end{aligned} \quad (60)$$

where $J_{\mathbf{q}S}$ contains the effect of the presence of external sources, to be specified in each particular case, and

$$\nu_{\mathbf{q}}^0 = \frac{1}{e^{\beta_0 \hbar \omega_{\mathbf{q}}} - 1},$$

is the distribution in equilibrium,

$$\tilde{\omega}_{\mathbf{q}} = \omega_{\mathbf{q}} + P_{\mathbf{q}}, \quad (61)$$

with

$$P_{\mathbf{q}} = \frac{\pi}{\hbar^2} \sum_{\mathbf{k}} |M_{\mathbf{k}\mathbf{q}}|^2 \frac{1 + \nu_{\mathbf{k}}^{TA} + \nu_{\mathbf{k}+\mathbf{q}}^{TA}}{\Omega_{\mathbf{k}+\mathbf{q}} + \Omega_{\mathbf{k}} - \omega_{\mathbf{q}}}, \quad (62)$$

which is the so-called self-energy correction, and

$$\Gamma_{\mathbf{q}} = \frac{\pi}{\hbar^2} \sum_{\mathbf{k}} |M_{\mathbf{kq}}|^2 (1 + \nu_{\mathbf{k}}^{TA} + \nu_{\mathbf{k}+\mathbf{q}}^{TA}) \delta(\Omega_{\mathbf{k}+\mathbf{q}} + \Omega_{\mathbf{k}} - \omega_{\mathbf{q}}) + \bar{\Gamma}_{\mathbf{q}}, \quad (63)$$

plays the role of the reciprocal of a relaxation time per mode, say $\tau_{\mathbf{q}}^{-1}$, (towards the equilibrium distribution), where

$$\nu_{\mathbf{k}}^{TA} = \frac{1}{e^{\beta_0 \hbar \Omega_{\mathbf{k}}} - 1}, \quad (64)$$

is the distribution in equilibrium at temperature T_0 of the TA phonons. In this Eq. (63) we have written explicitly the known contribution due to the anharmonic interaction, and in a Mathiessen rule [23] are also indicated in $\bar{\Gamma}_{\mathbf{q}}$ the sum of the inverse of the relaxation times due to the presence of impurities, imperfections, boundary conditions, etc.

Using Eq. (60) in Eq. (58), we obtain for the set of evolution equations that

$$\begin{aligned} \frac{\partial}{\partial t} I_p^{[\ell]}(\mathbf{r}, t) &= \sum_{\mathbf{q}} K_p^{[\ell]}(\mathbf{q}) [-\nabla \cdot \nabla_{\mathbf{q}} \tilde{\omega}_{\mathbf{q}}] \nu_{\mathbf{q}}(\mathbf{r}, t) \\ &\quad - \Gamma_{\mathbf{q}} [\nu_{\mathbf{q}}(\mathbf{r}, t) - \nu_{\mathbf{q}}^0] + J_{pS}^{[\ell]}(\mathbf{r}, t), \end{aligned} \quad (65)$$

for both families of sets (49) and (54), $p \equiv n$ and $p \equiv h$ respectively. A closure for the set of these equations must be introduced, that is, to express $\nu_{\mathbf{q}}(\mathbf{r}, t)$ that appears on the right hand side in terms of the hydrodynamic variables in the sets of Eqs. (49) and (54). First it can be noticed that Eq. (65) involves an enormous set of coupled nonlinear integro-differential equations, *i.e.*, both densities together with their fluxes of all orders are coupled through all these equations. To proceed further it is necessary to introduce a *contraction of description*, that is to say, to reduce the number of variables to be used, as described elsewhere [9,10,24]. Moreover, the two families in the sets of Eqs. (49) and (54) are coupled by cross-terms that account for thermo-striction effects. In cases where these effects are not particularly relevant they can be disregarded and we obtain two independent sets of evolution equations, one for the n -family and the other for the h -family.

Admitting that the just mentioned conditions are well satisfied, we introduce a contracted description of first order, and proceed to analyze the evolution equations for the energy density and its first flux. The resulting equations are [22]

$$\begin{aligned} \frac{\partial}{\partial t} h(\mathbf{r}, t) + \nabla \cdot \mathbf{I}_h(\mathbf{r}, t) &= a_L(t) \nabla \cdot \mathbf{I}_h(\mathbf{r}, t) \\ &\quad - \theta_h^{-1}(t) h(\mathbf{r}, t) + J_{hS}(\mathbf{r}, t), \end{aligned} \quad (66)$$

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{I}_h(\mathbf{r}, t) + \nabla \cdot I_h^{[2]}(\mathbf{r}, t) &= b_L^{[2]}(t) \cdot \nabla h(\mathbf{r}, t) \\ &\quad - \theta_I^{-1}(t) \mathbf{I}_h(\mathbf{r}, t) + \mathbf{J}_{IS}(\mathbf{r}, t), \end{aligned} \quad (67)$$

where

$$a_L(t) = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} \mathbf{a}_I(\mathbf{q}, t) \cdot \nabla_{\mathbf{q}} P_{\mathbf{q}}, \quad (68)$$

$$b_L^{[2]}(t) = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} a_h(\mathbf{q}, t) [\nabla_{\mathbf{q}} P_{\mathbf{q}} : \nabla_{\mathbf{q}} \omega_{\mathbf{q}}], \quad (69)$$

$$a_h(\mathbf{q}, t) = \hbar \omega_{\mathbf{q}} \nu_{\mathbf{q}}(t) [1 + \nu_{\mathbf{q}}(t)] \left[\sum_{\mathbf{q}} (\hbar \omega_{\mathbf{q}})^2 \nu_{\mathbf{q}}(t) [1 + \nu_{\mathbf{q}}(t)] \right]^{-1}, \quad (70)$$

$$\begin{aligned} \mathbf{a}_I(\mathbf{q}, t) &= \hbar \omega_{\mathbf{q}} \nu_{\mathbf{q}}(t) [1 + \nu_{\mathbf{q}}(t)] \left[\sum_{\mathbf{q}} (\hbar \omega_{\mathbf{q}})^2 \nu_{\mathbf{q}}(t) [1 + \nu_{\mathbf{q}}(t)] \times \right. \\ &\quad \left. [\nabla_{\mathbf{q}} \omega_{\mathbf{q}} \cdot \nabla_{\mathbf{q}} \omega_{\mathbf{q}}] \right]^{-1} \nabla_{\mathbf{q}} \omega_{\mathbf{q}}, \end{aligned} \quad (71)$$

$P_{\mathbf{q}}$ and $\Gamma_{\mathbf{q}}$ are given in Eqs. (62) and (63), and

$$\nu_{\mathbf{q}}(t) = \frac{1}{e^{F_{\mathbf{q}}(t)} - 1}, \quad (72)$$

with $F_{\mathbf{q}}$ of Eq. (48), we recall that

$$I_h^{[2]}(\mathbf{Q}, t) = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} [\nabla_{\mathbf{q}} \omega_{\mathbf{q}} : \nabla_{\mathbf{q}} \omega_{\mathbf{q}}] \nu_{\mathbf{q}}(\mathbf{Q}, t), \quad (73)$$

and J_{hS} and \mathbf{J}_{IS} are the contributions due to the coupling with the external sources that are present which are to be specified in each case being considered. We recall that $[\cdots : \cdots]$ stands for inner tensorial product.

Finally,

$$\theta_h^{-1}(t) = \sum_{\mathbf{q}} w_h(\mathbf{q}, t) \Gamma_{\mathbf{q}}, \quad (74)$$

and

$$\theta_I^{-1}(t) = \sum_{\mathbf{q}} w_I(\mathbf{q}, t) \Gamma_{\mathbf{q}}, \quad (75)$$

with

$$w_h(\mathbf{q}, t) = \frac{(\hbar \omega_{\mathbf{q}})^2 \nu_{\mathbf{q}}(1 + \nu_{\mathbf{q}})}{\sum_{\mathbf{q}'} (\hbar \omega_{\mathbf{q}'})^2 \nu_{\mathbf{q}'}(1 + \nu_{\mathbf{q}'})}, \quad (76)$$

$$w_I(\mathbf{q}, t) = \frac{(\hbar \omega_{\mathbf{q}})^2 \nu_{\mathbf{q}}(1 + \nu_{\mathbf{q}})}{\sum_{\mathbf{q}'} (\hbar \omega_{\mathbf{q}'})^2 \nu_{\mathbf{q}'}(1 + \nu_{\mathbf{q}'})} \frac{|\nabla_{\mathbf{q}} \omega_{\mathbf{q}}|^2}{|\nabla_{\mathbf{q}'} \omega_{\mathbf{q}'}|^2}, \quad (77)$$

which have the property of normalization, that is, it is verified that

$$\sum_{\mathbf{q}} w_h(\mathbf{q}, t) = 1, \quad (78)$$

and

$$\sum_{\mathbf{q}} w_I(\mathbf{q}, t) = 1, \quad (79)$$

Equations (74) and (75) define the reciprocal of Maxwell times associated to energy and its first flux, respectively, in this MHT of order 1. We can see that each Maxwell time follows a kind of Mathiessen rule: its inverse is a superposition of the reciprocal of the relaxation times of each mode $\tau_{\mathbf{q}}^{-1} = \Gamma_{\mathbf{q}}$, multiplied by a weighting function $w(\mathbf{q}, t)$. Because of the normalization condition (Eqs. (78) and (79)) it can be considered each $w(\mathbf{q}, t)$ as the probability at each time t of the contribution of the mode-relaxation time $\tau_{\mathbf{q}}$.

In a Debye model both Maxwell times are equal, *i.e.*,

$$\theta_h^{-1}(t) = \theta_I^{-1}(t) = \frac{\sum_{\mathbf{q}} (sq)^2 \nu_{\mathbf{q}} (1 + \nu_{\mathbf{q}}) \Gamma_{\mathbf{q}}}{\sum_{\mathbf{q}} (sq)^2 \nu_{\mathbf{q}} (1 + \nu_{\mathbf{q}})} \quad (80)$$

and we recall that $q \leq q_{Debye}$.

Taking into account that $|\nabla_{\mathbf{q}} \omega_{\mathbf{q}}| \simeq s$ for small values of q and that $|\nabla_{\mathbf{q}} \omega_{\mathbf{q}}| < s$ for intermediate to large values of q , we can estimate that in general $\theta_h > \theta_I$. It is conjectured that we would do have a hierarchy

$$\theta_h > \theta_I > \theta_{I_2} > \dots,$$

quite similar to the case of the classical fluid of previous sections.

V. MAXWELL TIMES IN MHT OF ELECTRONS IN SEMICONDUCTORS

We consider the case of a n-doped polar semiconductor, e.g., n-GaAs or n-GaN, whose Hamiltonian quantum mechanical operator is

$$\hat{H} = \hat{H}_{0e} + \hat{H}_{eph} + \hat{H}_{0ph} + \hat{H}_{e\varphi}, \quad (81)$$

where \hat{H}_{0e} is the Hamiltonian of the carriers (a concentration n of electrons in conduction band) taken in the effective mass approximation, with energy dispersion relation $\epsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m^*$. \hat{H}_{eph} is the Hamiltonian of the interactions of carriers and optical and acoustical phonons: we consider, in this case of polar semiconductors only the polar (Fröhlich) interaction with LO phonons, which is by large the predominant process of relaxation of the carriers to the lattice. \hat{H}_{0ph} is the Hamiltonian of the free LO phonons. $\hat{H}_{e\varphi}$ accounts for the interaction of the carriers with external pumping sources. We assume that the LO phonons remain in equilibrium with an external reservoir at temperature T_0 . The polarization effects (due the Coulomb interaction) modifying the mean field potential of the band states is not included, that is, we disregard the contribution from the plasmon states.

Indicating by $c_{\mathbf{k}}(c_{\mathbf{k}}^\dagger)$ the creation and annihilation operators in band states \mathbf{k} (we omit the spin index), as in the case of phonons of the previous Section we introduce as basic microdynamical variables for the corresponding NESEF the quantities

$$\{\hat{n}_{\mathbf{k}\mathbf{Q}} = c_{\mathbf{k}+\mathbf{Q}/2}^\dagger c_{\mathbf{k}-\mathbf{Q}/2}; \hat{H}_{0ph}\}. \quad (82)$$

For $\mathbf{Q} = 0$, $\hat{n}_{\mathbf{k}} = c_{\mathbf{k}}^\dagger c_{\mathbf{k}}$ is the operator occupation number in state \mathbf{k} , and the others with $\mathbf{Q} \neq 0$ accounts for variations in space. For the LO phonons we have taken \hat{H}_{0ph} once, as said, we consider them in constant equilibrium at temperature T_0 and therefore described by a canonical distribution. Hence, the statistical operator in NESEF of the electrons is

$$\varrho_\epsilon(t) = \exp \left\{ -\hat{S}(t, 0) + \int_{-\infty}^t dt' e^{\epsilon(t'-t)} \hat{S}(t', t'-t) \right\}, \quad (83)$$

where

$$\begin{aligned} \hat{S}(t_1, t_2) &= \ln \bar{\varrho}(t_1, t_2) = \\ &= \phi(t_1) + \sum_{\mathbf{k}} [F_{\mathbf{k}}(t_1) \hat{n}_{\mathbf{k}}(t_2) + \sum_{\mathbf{Q} \neq 0} F_{\mathbf{k}\mathbf{Q}}(t_1) \hat{n}_{\mathbf{k}\mathbf{Q}}(t_2)], \end{aligned} \quad (84)$$

where we recall that ϕ ensures the normalization of both $\bar{\varrho}$ and ϱ_ϵ , and $F_{\mathbf{k}}$ and $F_{\mathbf{kQ}}$ are the nonequilibrium thermodynamic variables conjugated to the basic ones $\hat{n}_{\mathbf{k}}$ and $\hat{n}_{\mathbf{kQ}}$ ($\mathbf{Q} \neq 0$). In Eq. (84) t_1 refers to the evolution in time of the nonequilibrium thermodynamic variables and t_2 to the evolution in time of the dynamical variables (Heisenberg representation).

Following the formalism we proceed to the derivation of the kinetic equation for the variable

$$n_{\mathbf{kQ}}(t) = \text{Tr}\{\hat{n}_{\mathbf{kQ}}\varrho_\epsilon(t)\}. \quad (85)$$

In the Markovian approximation after a calculation in reciprocal space and next going over to direct space it follows for

$$f_{\mathbf{k}}(\mathbf{r}, t) = \sum_{\mathbf{Q}} n_{\mathbf{kQ}}(t) e^{i\mathbf{Q} \cdot \mathbf{r}}, \quad (86)$$

that

$$\begin{aligned} \frac{\partial}{\partial t} f_{\mathbf{k}}(\mathbf{r}, t) + \frac{i}{\hbar} \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}} \cdot \nabla_{\mathbf{r}} f_{\mathbf{k}}(\mathbf{r}, t) = \\ \frac{2\pi}{\hbar} \sum_{\mathbf{q}} |\mathcal{C}_{\mathbf{q}}|^2 \{ [\nu_{\mathbf{q}}(t) + 1] f_{\mathbf{k}+\mathbf{q}}(\mathbf{r}, t) [1 - f_{\mathbf{k}}(\mathbf{r}, t)] - \\ \nu_{\mathbf{q}}(t) f_{\mathbf{k}}(\mathbf{r}, t) [1 - f_{\mathbf{k}+\mathbf{q}}(\mathbf{r}, t)] \} \delta(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}) - \\ \frac{2\pi}{\hbar} \sum_{\mathbf{q}} |\mathcal{C}_{\mathbf{q}}|^2 \{ [\nu_{\mathbf{q}}(t) + 1] f_{\mathbf{k}}(\mathbf{r}, t) [1 - f_{\mathbf{k}-\mathbf{q}}(\mathbf{r}, t)] - \\ \nu_{\mathbf{q}}(t) f_{\mathbf{k}-\mathbf{q}}(\mathbf{r}, t) [1 - f_{\mathbf{k}}(\mathbf{r}, t)] \} \delta(\epsilon_{\mathbf{k}-\mathbf{q}} - \epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}) \} + \\ J_{S\mathbf{k}}(\mathbf{r}, t), \end{aligned} \quad (87)$$

where $\mathcal{C}_{\mathbf{q}}$ is the matrix element of the Fröhlich electron-LO phonon interaction, $\omega_{\mathbf{q}}$ the frequency dispersion relation of the LO phonons, we recall that $\epsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m^*$, $J_{S\mathbf{k}}$ the interaction with external sources to be specified in each case, and $\nu_{\mathbf{q}}$ is the population of the LO phonons in equilibrium at temperature T_0 .

As in previous Sections we introduce the hydro-thermodynamic variables

$$I_p^{[\ell]}(\mathbf{r}, t) = \sum_{\mathbf{k}} K_p^{[\ell]}(\mathbf{k}) f_{\mathbf{k}}(\mathbf{r}, t), \quad (88)$$

$$K_n^{[\ell]}(\mathbf{k}) = \left[\frac{\hbar \mathbf{k}}{m^*} \dots \ell - \text{times} \dots \frac{\hbar \mathbf{k}}{m^*} \right], \quad (89)$$

$$K_h^{[\ell]}(\mathbf{k}) = \frac{\hbar^2 k^2}{2m^*} K_n^{[\ell]}(\mathbf{k}), \quad (90)$$

corresponding to the families of particle motion ($p \equiv n$) and energy motion ($p \equiv h$). The evolution equations are evidently given by

$$\frac{\partial}{\partial t} I_p^{[\ell]}(\mathbf{r}, t) = \sum_{\mathbf{k}} K_p^{[\ell]}(\mathbf{k}) \frac{\partial}{\partial t} f_{\mathbf{k}}(\mathbf{r}, t), \quad (91)$$

In what follows, for better visualization, we restrict the analysis to a description of order 2 of the heat transport introducing the reduced set of basic variables consisting of

$$\{h(\mathbf{r}, t); I_h(\mathbf{r}, t); I_{h2}(\mathbf{r}, t)\}, \quad (92)$$

where

$$h(\mathbf{r}, t) = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m^*} f_{\mathbf{k}}(\mathbf{r}, t), \quad (93)$$

$$\mathbf{I}_h(\mathbf{r}, t) = \sum_{\mathbf{k}} \frac{\hbar \mathbf{k}}{m^*} f_{\mathbf{k}}(\mathbf{r}, t), \quad (94)$$

and we separate the second order flux $I_h^{[2]}(\mathbf{r}, t)$ in the trace part and the remaining traceless part which is neglected. The trace of $I^{[2]}$ is

$$I_{h2}(\mathbf{r}, t) = \sum_{\mathbf{k}} \left(\frac{\hbar k}{m^*} \right)^2 f_{\mathbf{k}}(\mathbf{r}, t). \quad (95)$$

The auxiliary statistical operator is then given by

$$\begin{aligned} \bar{\varrho}(t, 0) = & \exp \left\{ -\phi(t, 0) \right. \\ & - \sum_{\mathbf{Q}} [F_h(\mathbf{Q}, t) \hat{h}(\mathbf{Q}) + \boldsymbol{\varphi}_h(\mathbf{Q}, t) \cdot \mathbf{I}_h(\mathbf{Q}, t) \\ & \left. + F_{h2}(\mathbf{Q}, t) I_{h2}(\mathbf{Q})] \right\}, \end{aligned} \quad (96)$$

which follows from Eq. (84) after taking, to be consistent with the contraction introduced, that

$$\begin{aligned} F_{\mathbf{kQ}}(t) = & \frac{\hbar^2 k^2}{2m^*} F_h(\mathbf{Q}, t) + \frac{\hbar^2 k^2}{2m^*} \hbar \mathbf{k} \cdot \boldsymbol{\varphi}_h(\mathbf{Q}, t) + \\ & \left(\frac{\hbar k}{m^*} \right)^2 F_{h2}(\mathbf{Q}, t). \end{aligned} \quad (97)$$

The equation of evolution for these hydro-thermodynamic variables in the absence of external sources are

$$\frac{\partial}{\partial t} h(\mathbf{r}, t) = -\nabla \cdot \mathbf{I}_h(\mathbf{r}, t) - \theta_h^{-1} h(\mathbf{r}, t) - a_0 I_{h2}(\mathbf{r}, t), \quad (98)$$

$$\frac{\partial}{\partial t} \mathbf{I}_h(\mathbf{r}, t) = -\nabla I_{h2}(\mathbf{r}, t) - \theta_{I_h}^{-1} \mathbf{I}_h(\mathbf{r}, t), \quad (99)$$

$$\frac{\partial}{\partial t} I_{h2}(\mathbf{r}, t) = -\frac{q}{m^* \beta} \nabla \cdot \mathbf{I}_h(\mathbf{r}, t) - a_2 h(\mathbf{r}, t) - \theta_{I_{h2}}^{-1} I_{h2}(\mathbf{r}, t). \quad (100)$$

The expression for the trace of the divergence of the third order flux in terms of the basic macrovariables, namely

$$\text{Tr}\{\nabla \cdot I_h^{[3]}(\mathbf{r}, t)\} = \frac{q}{m^* \beta} \nabla \cdot \mathbf{I}_h(\mathbf{r}, t) - a_2 h(\mathbf{r}, t), \quad (101)$$

was obtained resorting to Heims-Jaynes perturbation expansion for averages [25] around the homogeneous state where

$$a_2 = \frac{1}{m^* \beta \theta_{I_2}}, \quad (102)$$

after using that

$$f_{\mathbf{k}} = \frac{1}{\exp\{\beta \hbar^2 k^2 / 2m^*\} + 1}, \quad (103)$$

$$\beta = \frac{1}{k_B T^*}, \quad (104)$$

with T^* being the nonequilibrium temperature (quasitemperature) of the electrons, and in Eq. (84) making the identification

$$F_{\mathbf{k}} \equiv \frac{\hbar^2 k^2}{2m^*} \beta(t), \quad (105)$$

In Eqs. (98) to (100) are present the Maxwell times θ_h , θ_{I_h} and $\theta_{I_{h2}}$, associated to the energy, its vectorial (first) flux (current of heat) and the second flux respectively. They are given by

$$\tau_h^{-1} = \frac{14}{5} \xi \tau_0^{-1}, \quad (106)$$

$$\tau_{I_h}^{-1} = \frac{2}{25} \xi \tau_h^{-1} = \frac{2}{25} \frac{14}{5} \xi^2 \tau_0^{-1}, \quad (107)$$

$$\tau_{I_{h2}}^{-1} = \frac{1}{28} \xi^2 \tau_h^{-1} = \frac{1}{25} \frac{14}{5} \xi^3 \tau_0^{-1}, \quad (108)$$

where

$$\xi = \beta \hbar \omega_0 = \frac{\hbar \omega_0}{k_B T^*}, \quad (109)$$

and

$$\tau_0^{-1} = , \quad (110)$$

in the calculation being used the parameters corresponding to GaAs given in Table I.

TABLE I: Parameters characteristic of GaAs

Parameter	Value
electron effective mass, m^*	$0.067m_0$
static dielectric constant, ε_0	12.91
optical dielectric constant, ε_∞	10.91
LO phonon energy, $\hbar\omega_{LO}$	37.0 meV
mass density, ρ	5.31 g/cm ³

Accordingly, it follows that the comparison of the three values follows from

$$\frac{\tau_{I_h}}{\tau_h} = \frac{875}{14} \frac{k_B T^*}{\hbar\omega_0}, \quad (111)$$

$$\frac{\tau_{I_{h2}}}{\tau_h} = 28 \left(\frac{k_B T^*}{\hbar\omega_0} \right)^2, \quad (112)$$

$$\frac{\tau_{I_h}}{\tau_I} = \frac{56}{125} \frac{k_B T^*}{\hbar\omega_0}, \quad (113)$$

and, therefore, their hierarchy depends on the ratio $\hbar\omega_0/k_B T^*$, that is, on the ratio of the energy of LO-phonon modes to the nonequilibrium thermal kinetic energy of the carriers. Numerical values of the three Maxwell times for several values of the carriers' quasitemperature are given in Table II.

TABLE II: Numerical Values of Maxwell Times for GaAs

T^* (K)	θ_h (ps)	θ_{I_h} (ps)	$\theta_{I_{h2}}$ (ps)
100	23.7	33.76	193
150	3	6.4	36.5
200	1.24	3.5	20
300	0.57	2.45	19
500	0.43	3.06	17.5
1000	0.43	6.12	3.5

It can be noticed that these Maxwell times are in order of picoseconds to femtoseconds. We stress that T^* is the carriers' quasitemperature, that is, the nonequilibrium temperature which is a measure in Kelvin degrees of their nonequilibrium energy [26].

Inspection of Table II tells us that it is verified that $\theta_h < \theta_{I_h} < \theta_{I_{h2}}$, with this ordering being inverted for values of ξ sufficiently larger than 1, i.e., for carriers' quasitemperatures sufficiently larger than Einstein temperature ($T_E = \hbar\omega_0/k_B$) of the LO phonons, roughly $T^* > 432$ Kelvin in GaAs.

VI. CONCLUDING REMARKS

In the framework of Mesoscopic Hydro-Thermodynamics applied to the study of the hydrodynamic motion of a classical molecular fluid, and of the "fluids" of phonons and carriers in intrinsic and doped semiconductors, we have characterized the so-called Maxwell times, or, better to say, generalizations of Maxwell original proposal [1] as described in the Introduction. These Maxwell times are associated with the fluxes of all orders present in MHT (fluxes of particles – or quasi-particles – and of heat). A complete characterization is obtained in the case of a classical fluid (of molecules, polymers, etc.), presented in Section II and III. It follows the quite important result that they satisfy a hierarchy showing that they are increasingly diminishing as the order of the fluxes increases. This allows for establishing criteria for the choice of the contraction of description to be used in each case.

In other words to introduce an appropriate – for each case – contraction of description: *this contraction implies in retaining the information considered as relevant for the problem in hands, and to disregard irrelevant information* [27].

Elsewhere [24] it has been discussed the question of the contraction of description (reduction of the dimensions of the nonequilibrium thermodynamic space of states), where a criterion for justifying the different levels of contraction is derived: It depends on the range of wavelengths and frequencies which are relevant for the characterization, in terms of normal modes, of the hydro-thermodynamic motion in the nonequilibrium open system. It can be shown that the truncation criterion *rests on the characteristics of the hydrodynamic motion that develops under the given experimental procedure*.

Inclusion of higher and higher-order fluxes implies in describing a motion involving increasing Knudsen numbers per hydrodynamic mode, that is, governed by smaller and smaller wavelengths – larger and larger wavenumbers – accompanied by higher and higher frequencies. In a qualitative manner, we can say that, as a general "thumb rule", the criterion indicates that *a more and more restricted contraction can be used when larger and larger are the prevalent wavelengths in the motion*. Therefore, in simpler words, when the motion becomes more and more smooth in space and time the more reduced can be the dimension of the space of basic macrovariables to be used for the description of the nonequilibrium thermodynamic state of the system. It can be conjectured a general criterion for contraction, namely, a contraction of order r (meaning keeping the densities and their fluxes up to

order r) can be introduced once we can show that in the spectrum of wavelengths, which characterize the motion, predominate those larger than a “frontier” one, $\lambda_{(r,r+1)}^2 = v^2 \theta_r \theta_{r+1}$ where v is of the order of the thermal velocity and θ_r and θ_{r+1} the corresponding *Maxwell times* associated to the r and $r + 1$ order fluxes.

Section IV has been devoted to the characterization of Maxwell times in the thermal motion of phonons in intrinsic semiconductors. It is shown the interesting results that the Maxwell times have an expression for their inverse in the form of a weighted Mathiessen-like rule involving the inverse of the relaxation time of all phonon modes. We can say that it is expected once the hydrodynamic motion is a collective one composed of the contribution of all the phonon modes. As in the previous case Maxwell times follow a hierarchy involving ever decreasing values as the order of the fluxes increases.

Finally, in Section V, for the case of doped polar semiconductors, the Maxwell times associated to the hydrodynamic motion of the electrons in Bloch bands have been obtained. They are mainly determined by the presence of the polar Fröhlich interaction between electrons and phonons and they follow a hierarchy of values, which is dependent on the ratio of the phonon energy $\hbar\omega_0$ with the electron thermal (kinetic) energy: it implies in a series of decreasing values as long as the quasitemperature of the “hot” electrons is higher than the Einstein temperature of the phonons.

Acknowledgments: The authors would like to acknowledge partial financial support received from the São Paulo State Research Agency (FAPESP) and the Goiás State Research Agency (FAPEG).

In Memoriam: *With very sad feelings, we regret to report the passing away of our dear colleague Áurea Rosas Vasconcellos, a genuine, devoted and extremely competent Teacher and Researcher with fervent dedication to Theoretical Physics in the Condensed Matter area. She had a very important contribution to this paper, particularly in what refers to the “quantum” fluids of phonons and electrons in semiconductors.*

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